Repeated Games with Bounded Memory and the Entropy Method

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1 Warm up: the entropy method

Shannon's entropy of a discrete random variable x is defined as

$$H(x) = -\sum_{\xi} \mathbf{P}(x = \xi) \log(\mathbf{P}(x = \xi)).$$

The entropy of a Bernoulli random variable with parameter p is denoted $H(p)$. The mutual information of two random variables is defined as $I(x; y) = H(x) + H(y) - H(x, y)$.

Exercise 1. $\binom{n}{k} \leq e^{nH\left(\frac{k}{n}\right)}$.

Exercise 2. If $k \leq \frac{n}{2}$, then $\sum_{i=0}^{k} \binom{n}{i} \leq e^{nH\left(\frac{k}{n}\right)}$.

More generally, let A be a finite alphabet. Denote the cardinality of A by $|A|$, the set of all probability measures over A by $\Delta(A)$, and the set of all n-periodic A-sequences by $A^{(n)} = \{x \in A^{\mathbb{Z}} : s = t \mod n \to x_s = x_t\}$, and the empirical frequency of $x \in A^{(n)}$ by $\mathrm{emp}(x) \in \Delta(A)$.

Exercise 3. For every $X \subset A^{(n)}$, $|X| \leq e^{n \max\{H(p) : p \in \mathrm{conv}\{\mathrm{emp}(x) : x \in X\}\}}$.

Define $A^{(n,k)} \subset A^{(n)}$ as the set of all n-periodic A-sequences that do not contain any k-subsequence more than once in the same period. Formally,

$$A^{(n,k)} = \{x \in A^{\mathbb{Z}} : (x_{s+1}, \ldots, x_{s+k}) = (x_{t+1}, \ldots, x_{t+k}) \leftrightarrow s = t \mod n\}.$$

Exercise 4. For every n, k, and $x \in A^{(n,k)}$, $n \leq e^{kH(\mathrm{emp}(x))}$.

The next problem seems too difficult to be called an "exercise" (at least I'm not aware of a simple argument that proves it).

Proposition 5. *Let $x_1, x_2, \ldots$ be i.i.d. A-valued random variables. Let $T_k = \min\{n : (x_1, \ldots, x_n)^\omega \notin A^{(n,k)}\}$. Then,*

$$\sup\{r > 0 : \lim_{k\to\infty} \mathbf{P}(T_k > r^k)^{\frac{1}{r^k}} = 1\} = e^{H(x_1)}.$$

2 Game definition

2.1 One-stage game

In this exposition a one-stage game $G = (I, A, g)$ consists of the following components:

- a finite set of players $I = \{1, \dots, n\}$,
- finite sets of actions, $A_1, \dots, A_n$, $A = A_1 \times \cdots \times A_n$,
- a payoff function (of Player 1) $g : A \to \mathbb{R}$.

The payoff function extends to $g : \Delta(A) \to \mathbb{R}$ linearly. The min max value of (Player 1 in) G is defined as

$$\min\max G = \min_{\substack{p_j \in \Delta(A_j) \\ j=2,\dots n}} \max_{p_1 \in \Delta(A_1)} g(p_1 \otimes \cdots \otimes p_n).$$

2.2 Repeated game

A pure strategy for Player i in the repeated version of G is a function $\sigma_i \colon \bigcup_{t=0}^{\infty} A^t \to A_i$. A profile of strategies $\sigma_1, \dots, \sigma_n$ induces an infinite play $a_1, a_2 \dots \in A$ defined recursively by

$$a_{t+1} = (\sigma_1(a_1, \dots, a_t), \dots, \sigma_n(a_1, \dots, a_t)).$$

The (limiting average) payoff of the repeated version of G is defined as

$$g_*(\sigma_1, \dots, \sigma_n) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} g(a_t),$$

wherever the limit exists (we will only consider situations where the limit exists).

2.3 Repeated game with bounded recall

A k-recall strategy is a strategy that relies only on the last k stages of history. Formally, σ_i is a k-recall strategy if

$$\sigma_i(a_1, \dots, a_t) = \sigma_i(a_{t-k+1}, \dots, a_t),$$

for every $t > k$ and $a_1, \dots, a_t \in A$. The set of k-recall strategies for player i is denoted $\Sigma_i(k)$. The repeated game with recall capacities $k_1, \dots, k_n \in \mathbb{N}$ is the game

$$G[k_1, \dots, k_n] = (I, \Sigma_1(k_1) \times \cdots \times \Sigma_n(k_n), g_*).$$

The holy grail is a full characterization of $\min\max G[k_1,\dots,k_n]$. The case of two players is fairly well understood (asymptotically, for large k_i). However, our understanding of games with more than two players is limited.

3 Two-player games

Theorem 6 (P. (2012)). *For every $C>0$, $C\neq\log(|A_1|)$,*

$$\lim_{k\to\infty}\min\max G[k,e^{Ck}] = \max_{\substack{p_1\in\Delta(A_1)\\ H(p_1)\geq C \text{ or}\\ H(p_1)=0}}\ \min_{p_2\in\Delta(A_2)} g(p_1\otimes p_2).$$

An inequality in the direction "$\geq$" stems from Proposion 5 and the following potent lemma (whose proof is a nice exercise).

Lemma 7 (Neyman and Okada (2009)). *Let $x_1,\dots,x_n,y_0,\dots,y_n$ be (discrete) random variables such that y_k is measurable w.r.t. $\sigma\langle x_i,y_i : i<k\rangle$, for all $k=1,\dots,n$. Let t be a random variable uniformly distributed in $\{1,\dots,n\}$ independently of the other random variables. Then,*

$$I(x_t;y_t)\leq H(x_t)-\frac{1}{n}H(x_1,\dots,x_n)+\frac{1}{n}I(x_1,\dots,x_n;y_0).$$

[recall: $I(x;y)$ measures the interdependence between x and y.]

Hint. To prove Theorem 6, fix some time T and let y_0 be the continuation strategy of Player 2 at time T, and (x_i,y_i) the induced play at time $T+i$. One needs to set n large enough so that $\frac{1}{n}H(y_0)$ vanishes, but not too large so that Player one can implement a (stationary) play whose n-stage average entropy is close to its one-stage entropy (i.e., $H(x_t)-\frac{1}{n}H(x_1,\dots,x_n)$ vanishes).

The critical point $C=\log(|A_1|)$

The expression on the RHS in Theorem 6 is a non-increasing function of C which is continuous at all but perhaps one point, $C=\log(|A_1|)$. Let $\bar{v}$, $\underline{v}$ be the left and right limits at $C=\log(|A_1|)$ respectively. P. (2012) explains how to extend the proof of Theorem 6 to showing that $\lim_{k\to\infty}\min\max G[k,(12k)^{-1}|A_1|^k]=\bar{v}$. I believe that one can also extend the proof to showing that $\lim_{k\to\infty}\min\max G[k,k^D|A_1|^k]=\underline{v}$, for some (sufficiently large) constant D. The following is an open research problem.

Problem 8. $\lim_{k\to\infty}\min\max G[k,|A_1|^k]=?$ *Does the limit necessarily exits?*

Conjecture 9.

$$\lim_{C\to 0^+} \liminf_{k\to\infty} \min\max G[k, C|A_1|^k] = \min\max G,$$

$$\lim_{C\to\infty} \limsup_{k\to\infty} \min\max G[k, C|A_1|^k] = \max_{x\in A_1} \min_{y\in A_2} g(x,y).$$

4 Three players (or more)

Estimating $\min\max G[k_1, k_2, k_3]$ seems to be difficult. The difficulty stems from the fact that the set over which we minimize is not convex. We don't even know how to deal with natural special cases such as $k_1 = k_2 = k_3$. It seems as though the interesting region is where the k_i-s are proportional to each other. From Theorem 6 it isn't hard to show that for every $C > 0$,

$$\min_{p_{2,3}\in\Delta(A_2\times A_3)} \max_{p_1\in\Delta(A_1)} g(p_1\otimes p_{2,3}) - o(1) \le \min\max G[k, Ck, Ck] \le \min\max G + o(1).$$

It is not too hard to show that if the recall capacity of Players 2 and 3 is much larger than that of Player 1, then Players 2 and 3 can correlate against Player 1 . Formally, for any game G,

$$\lim_{C\to\infty} \limsup_{k\to\infty} \min\max G[k, Ck, Ck] = \min_{p_{2,3}\in\Delta(A_2\times A_3)} \max_{p_1\in\Delta(A_1)} g(p_1 \otimes p_{2,3}).$$

Surprisingly, Players 2 and 3 can sometimes correlate even if their recall capacity is smaller than that of Player 1.

Theorem 10. *(P. 2013) For every $\epsilon > 0$ and every game G, by cloning any of the actions of Player 2 sufficiently many times we obtain a game $\tilde{G}$ such that*

$$\min\max \tilde{G}[k, \epsilon k, \epsilon k] \le \min_{p_{2,3}\in\Delta(A_2\times A_3)} \max_{p_1\in\Delta(A_1)} g(p_1 \otimes p_{2,3}) + \epsilon.$$

For a fixed game, we have a converse result.

Theorem 11 (Bavly and P. 2018+)**.** *For every game G,*

$$\lim_{\epsilon\to 0^+} \liminf_{k\to\infty} \min\max G[k, \epsilon k, \epsilon k] = \min\max G.$$

We don't have good estimates for $\min\max G[k, Ck, Ck]$, given a game G and a constant $C > 0$.

Conjecture 12. $\lim_{k\to\infty} \min\max G[k, Ck, Ck]$ *exists, for every one-stage game G and every $C > 0$.*