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The Lipschitz constant of perturbed anonymous games

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Abstract
The Lipschitz constant of a game measures the maximal amount of influence that one player has on the payoff of some other player. The worst-case Lipschitz constant of an n-player k-action δ-perturbed game, $\lambda(n, k, \delta)$, is given an explicit probabilistic description. In the case of $k \geq 3$, it is identified with the passage probability of a certain symmetric random walk on $\mathbb{Z}$. In the case of $k = 2$ and n even, $\lambda(n, 2, \delta)$ is identified with the probability that two i.i.d. binomial random variables are equal. The remaining case, $k = 2$ and n odd, is bounded through the adjacent (even) values of n. Our characterization implies a sharp closed-form asymptotic estimate of $\lambda(n, k, \delta)$ as $\delta n/k \to \infty$.

Keywords Anonymous games · Large games · Perturbed games · Approximate Nash equilibrium

1 Introduction

The Lipschitz constant of a game measures the maximal amount of influence that one player has on the payoff of some other player. Identifying classes of games that admit a small Lipschitz constant is important due to the stability and robustness of their equilibria (Kalai 2004; Azrieli and Shmaya 2013; Al-Najjar and Smorodinsky 2000). The Lipschitz constant is given an explicit description in the class of perturbed anonymous games (see Theorems 1, 2, and 3).

Schmeidler (1973) taught us that games with a continuum of anonymous players always admit a Nash equilibrium in pure strategies. Since a continuum of players is an idealization of a large finite set of players, it is reasonable to believe that large finite anonymous games should admit an approximation of a pure Nash equilibrium

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of some sort. Of what sort and how fast does this approximation emerge (as the number of players grows)? These questions are given precise answers[1] in Theorem 4.

Before explaining our notion of approximation let us start with a naïve attempt and then see why it does not work. Perhaps the existence of a pure Nash equilibrium in non-atomic anonymous games suggests the existence of an approximate (ϵ-) Nash equilibrium in large finite anonymous games? The answer is "no." Let's see why. Consider a game in which the players are people who decide whether to go to a party or not. For some reason some of the people prefer parties with an even number of participants while others prefer an odd number. This game is anonymous, since the players don't care about the identity of the party participants but only about their number. Alas, this game does not admit any pure Nash equilibrium, not even an approximate one, regardless of the number of players. The instability of this game stems from the persistence of its Lipschitz constant. The influence of a single player on another player's payoff remains the same regardless of the number of players.

The notion that does do the trick is that of an approximate Nash equilibrium in *perturbed* pure strategies. A perturbed pure strategy is a deviation from a pure strategy to the uniformly mixed strategy with some (small $\delta > 0$) probability. Assume that all of the players in our example play perturbed pure strategies. It is now clear that the size of the game matters. When the number of players is small, it is likely that none of the players will play randomly and therefore there is no approximate equilibrium. However, as the number of players grows, it becomes more and more likely that at least one of the players will randomize and therefore all the players will become almost indifferent whether to go to the party or not; and therefore a pure approximate Nash equilibrium exists (in fact, any perturbed pure strategy profile will constitute an approximate Nash equilibrium).

The trick of perturbing all players' actions works for anonymous games generally. The rigorous explanation relies on analysis of the Lipschitz constant of the perturbed game. The accumulative effect of many small perturbations is the reduction of the Lipschitz constant of the game and, thus, the emergence of a pure approximate Nash equilibrium, which translates to a perturbed pure approximate Nash equilibrium in the original (unperturbed) game.

Given parameters n, k, and δ, we give an explicit expression for the worst-case (largest) Lipschitz constant of any n-player k-action δ-perturbed anonymous game. The expression is given in terms of a symmetric random walk on the integers. For $k \geq 3$, the expression is the tail probability of the first passage time (from 0 to 1). For $k = 2$ and n is even, the expression is the probability that a certain random walk lands at 0 at time $n/2 - 1$. When n is odd, we don't have an exact expression, only upper and lower bounds that use the adjacent (even) values of n.

The Lipschitz constant of perturbed anonymous games has algorithmic applications, as well. Goldberg and Turchetta (2017) presented an efficient algorithm for computing approximate Nash equilibrium in n-player two-action anonymous games. Their algorithm relies on the existence of an approximate equilibrium that uses perturbed pure strategies. The existence of such an equilibrium is guaranteed (due to Azrieli and

[1] Recent papers that address these questions include Gradwohl and Reingold (2010), Deb and Kalai (2015), and Guilherme and Konrad (2020).

Shmaya 2013) since perturbed anonymous games admit a small Lipschitz constant. The premise of the method of Goldberg and Turchetta (2017) depends on how tightly one estimates the Lipschitz constant of the perturbed game. Goldberg and Turchetta (2017) obtained an inverse polynomial upper-bound (in n, the number of players, assuming 2 actions for each player) that enabled them to prove that their algorithm was polynomial. Cheng et al. (2017) improved the upper bound and extended it to any number of actions, k, showing that the Lipschitz constant is $\tilde{\mathcal{O}}\left(\sqrt{k^9(\delta n)^{-1}}\right)$. We provide an asymptotically sharp approximation for the worst-case Lipschitz constant $\lambda = \lambda(n, k, \delta)$ by identifying it with a passage time of a certain symmetric random walk on $\mathbb{Z}$. For example, our characterization implies that $\lambda = \mathcal{O}\left(\sqrt{k(\delta n)^{-1}}\right)$, as $\delta + k(\delta n)^{-1} \to 0$.

Our contribution. Schmeidler (1973) showed that any non-atomic anonymous game admits a pure Nash equilibrium. Kalai (2004) proved an analogous result for large finite games. Specifically, Kalai (2004) showed that if the game is (i) anonymous, (ii) has a large number of players, and (iii) the influence of each player on the payoffs of the other players is small, then the game admits a pure approximate Nash equilibrium. Azrieli and Shmaya (2013) proved a similar result assuming only (ii) and (iii), (namely, the game need not be anonymous) coining also the notion of the "Lipschitz constant" of a game. In addition, Azrieli and Shmaya (2013) obtained an even better approximation under assumption (i). Goldberg and Turchetta (2017) relaxed assumption (iii) while keeping assumptions (i) and (ii). Namely, they assumed anonymity and a large number of players, but not a small Lipschitz constant. Goldberg and Turchetta (2017) used perturbation to obtain approximate Nash equilibria in binary-action anonymous games. Cheng et al. (2017) generalized Goldberg and Turchetta's result to k-action games. Both Goldberg and Turchetta (2017) and Cheng et al. (2017) rely for their results on upper-bounds for $\lambda(n, k, \delta)$. Our characterization of $\lambda(n, k, \delta)$ improves upon the previous best known upper bound and constitutes the first obtained lower bound.

Our main contribution is to explore what we believe is the *correct* point of view at the problem that allows us to obtain an exact characterisation of $\lambda(n, k, \delta)$ while resorting to elementary tools only. We embed a certain random walk on $\mathbb{Z}$ in the probability space defined by a coupling of two adjacent perturbed pure strategy profiles. The distance between the payoffs of the two profiles is bounded from above by the probability that the passage time (from 0 to 1) of the random walk is at least $n - 2$. We apply the reflection principle of symmetric random walks to show that our upper bound is exact in the case $k \geq 3$. In the case $k = 2$ and n even we use a convexity argument to show that the worst scenario occurs when half of the players take one action the other half take the other action. For $k = 2$ and n odd we find that the $\lambda_n := (n, 2, \delta)$ lies between λ_{n+1} and the geometric mean of λ_{n-1} and λ_{n+1}.

2 Definitions and results

2.1 Lipschitz constant

An n-player k-action game[2] is a function $g\colon [k]^n \to [0,1]^n$. Following Azrieli and Shmaya (2013), the Lipschitz constant of a game is the maximal change in some player's payoff when a single opponent changes his strategy.

Formally, the Hamming distance between two pure strategy profiles $a, b \in [k]^n$ is defined as

$$\rho(a,b) = |\{i \in [n] : a_i \neq b_i\}|.$$

The Lipschitz constant of g is defined as

$$\lambda(g) = \max |g_i(a) - g_i(b)|,$$

where the maximum is over all $i \in [n]$ and $a, b \in [k]^n$ such that $a_i = b_i$ and $\rho(a,b) = 1$.

2.2 Perturbation

For $0 < \delta < 1$, the δ-perturbation[3] of a strategy $a_i \in [k]$ is the following mixture of a_i and the uniform distribution $u \sim \text{uniform}([k])$,

$$a_i^\delta = (1-\delta)a_i + \delta u.$$

The δ-perturbation of g is the game $g^\delta\colon [k]^n \to [0,1]^n$ defined by

$$g^\delta(a_1,\ldots,a_n) = E\left[g(a_1^\delta,\ldots,a_n^\delta)\right].$$

2.3 Anonymous games

A game g is called *anonymous* if, for every $i \in [n]$, $g_i(\cdot)$ is a function of i's own action and the number of other players who take each action $j \in [k]$. Formally, g is anonymous if $g_i(a) = g_i(b)$, for every $i \in [n]$ and every $a, b \in [k]^n$ such that $a_i = b_i$ and $|\{i' \in [n] : a_{i'} = j\}| = |\{i' \in [n] : b_{i'} = j\}|$, for every $j \in [k]$.

[2] Restricting the discussion to games with payoffs in [0, 1] does not cause a loss of generality, since the games considered are finite games whose payoffs can be normalized to be in [0, 1] through an affine transformation. The influence of such an affine transformation on the Lipschitz constant of the game is multiplication by the Lipschitz constant of the transformation.

[3] The notion of perturbed strategies in not knew. It appeared in other contexts in game theory, for example, in the definition of Selten's trembling hand perfect equilibrium (Selten 1975).

2.4 Symmetric random walk on the integers

The statement of our first result uses the notion of a symmetric random walk on $\mathbb{Z}$ with (stationary) rate r, which is a sequence of random variables, $S_0^r, S_1^r, \ldots$, whose law is defined by[4]

$$P(S_0^r = 0) = 1,$$
$$P(S_{n+1}^r - S_n^r = 0 | S_n^r) = 1 - r,$$
$$P(S_{n+1}^r - S_n^r = +1 | S_n^r) = P(S_{n+1}^r - S_n^r = -1 | S_n^r) = \tfrac{r}{2}.$$

2.5 Our results

Our objective is to characterize the worst-case Lipschitz constant of anonymous games defined by

$$\lambda(n, k, \delta) = \max_g \lambda(g^\delta),$$

where the maximum is over all n-player k-action anonymous games.

For games with $k \geq 3$ actions we obtain the following characterization.

Theorem 1 *For every $n \geq 2$, $k \geq 3$, and $\delta \in (0, 1)$,*

$$\lambda(n, k, \delta) = (1 - \delta) P(S_{n-2}^{2\delta/k} \in \{0, 1\}).$$

For games with two actions we have an exact characterization when the number of players is even and an estimation when it is odd.

Theorem 2 *For every $n \in \mathbb{N}$, and $\delta \in (0, 1)$, let us abbreviate $\lambda_n = \lambda(n, 2, \delta)$. Then,*

$$\lambda_{2n} = (1 - \delta) P\left(S_{n-1}^{\delta(1-\delta/2)} = 0\right),$$

and

$$\lambda_{2n+1} \in \left[\lambda_{2n+2}, \sqrt{\lambda_{2n}\lambda_{2n+2}}\right].$$

We obtain the following asymptotically sharp approximation[5] for the case where n is large relative to k and δ^{-1}.

Theorem 3 *For $k \geq 3$,*

$$\lim_{\frac{n\delta}{k} \to \infty} (1 - \delta)^{-1} \sqrt{\frac{\pi n \delta}{k}} \times \lambda(n, k, \delta) = 1.$$

[4] A referee noted that S_n^r is the sum of n i.i.d. random variables that take the value zero with probability $1 - r$ and the values plus and minus one with probability $\frac{r}{2}$.

[5] An example of a game that attains the asymptotic expression in the case $k = 3$ appears in (Al-Najjar and Smorodinsky 2000, p. 323).

For $k = 2$,

$$\lim_{n\delta \to \infty} (1-\delta)^{-1}\sqrt{\pi n \delta(1-\delta/2)} \times \lambda(n, 2, \delta) = 1.$$

The following theorem says that anonymous games with a large number of players admit an approximate Nash equilibrium in perturbed pure strategies.

Theorem 4 *Every n-player k-action game admits an ϵ-Nash equilibrium in δ-perturbed pure strategies, whenever $\epsilon \geq \delta + 2k\lambda(n, k, \delta)$.*

Furthermore, there exist functions $\epsilon(n, k), \delta(n, k) = \mathcal{O}(kn^{-\frac{1}{3}})$, such that every n-player k-action game admits an $\epsilon(n, k)$-Nash equilibrium in $\delta(n, k)$-perturbed pure strategies.

3 Preliminaries

3.1 The reflection principle

A symmetric random walk on $\mathbb{Z}$ is a sequence of random variables, $S_1, S_2 \ldots$, such that the increments $I_i := S_i - S_{i-1}$ (where $S_0 := 0$) satisfy

- $I_1, I_2, \ldots \in \{0, 1, -1\}$,
- $I_1, I_2, \ldots$ are mutually independent,
- $E[I_i] = 0$, for all i.

We will use the following property of symmetric random walks.[6]

Lemma 5 (Reflection Principle) *Let $S_1, \ldots, S_n$ be a symmetric random walk on $\mathbb{Z}$; then*

$$P(S_1 < 1, \ldots, S_n < 1) = P(S_n \in \{0, 1\}).$$

Proof Let $T = \min\{t \in \mathbb{N} : S_t = 1\}$. The event $\{T \leq n\}$ is the complement of the event $\{S_1 < 1, \ldots, S_n < 1\}$, and

$$\begin{aligned} P(T \leq n) &= P(S_n > 1, T \leq n) + P(S_n < 1, T \leq n) + P(S_n = 1, T \leq n) \\ &= 2P(S_n > 1, T \leq n) + P(S_n = 1, T \leq n) = 2P(S_n > 1) + P(S_n = 1) \\ &= P(S_n > 1) + P(S_n < -1) + P(S_n = 1) \\ &= P(S_n \notin \{-1, 0\}) = P(S_n \notin \{0, 1\}). \end{aligned}$$

□

[6] The reflection principle has become folklore in the theory of random walks. It is often attributed to the French mathematician Désiré André, who used it slightly differently than the way we do here. Lemma 5 is very similar to Lemma 3.3.1 in (Feller 1968, p. 76).

3.2 The total variation distance

Let X and Y be random variables assuming values in a finite or countable infinite set A. The total variation distance between (the distributions of) X and Y is denoted by $d_{TV}(X, Y)$. The following definitions are equivalent:

$$\begin{aligned} d_{TV}(X, Y) &= \inf_{X' \sim X,\ Y' \sim Y} P(X' \neq Y') \\ &= \max_{f:\ A \to [0,1]} \mathbb{E}[f(X)] - \mathbb{E}[f(Y)] \\ &= \sum_{a \in A} [P(X = a) - P(Y = a)]_+ \\ &= \tfrac{1}{2} \sum_{a \in A} |P(X = a) - P(Y = a)|\,. \end{aligned}$$

3.3 The Poisson binomial distribution

A standard Poisson binomial random variable is a finite sum of independent (not necessarily identically distributed) Bernoulli random variables. We define a Poisson binomial (PB) random variable as the sum of a standard Poisson binomial random variable and an integer. Note the if X and Y are independent PB random variables, then $X + Y$ and $X - Y$ are PB as well. The distribution of a PB random variable is called a PB distribution.

A PB distribution is unimodal and its mode is attained at the mean up to rounding to a nearby integer (see Samuels 1965). It follows that if X is a PB random variable, then the total variation distance between X and $X + 1$ is the value of X at its mode. We will use the following conclusion.

Lemma 6 *Let X be a PB random variable with $\mu = E[X]$. We have,*

$$d_{TV}(X, X + 1) = \max_{t \in \mathbb{Z}} P(X = t) = \max\{P(X = \lfloor \mu \rfloor), P(X = \lceil \mu \rceil)\}.$$

A PB distribution with a large variance can be approximated by a normal distribution with the same mean and variance in a very strong sense. Let $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ be the the standard normal density. The following lemma is taken from Pitman (Pitman 1997, Eq. (25)) who attributes it to Platonov (1980).

Lemma 7 *Let X be a PB random variable with $\mu = E[X]$, and $\sigma^2 = Var[X]$. For every $t \in \mathbb{Z}$,*

$$\left| \sigma P(X = t) - \phi\left(\frac{t - \mu}{\sigma}\right) \right| \leq \frac{C}{\sigma},$$

for some global constant C.

4 Proofs

We denote the indicator vector of an action $j \in [k]$ by $e_j \in \mathbb{R}^k$. For a strategy profile $a = (a_1, \ldots, a_n) \in [k]^n$, define

$$N(a) = \sum_{i=1}^{n} e_{a_i}.$$

Namely, $N(a) \in \mathbb{Z}_+^k$ is the vector that counts the number of players who take each one of the actions. Since a perturbed action profile $a^\delta = (a_1^\delta, \ldots, a_n^\delta)$ is a random variable that takes values in $[k]^n$, $N(a^\delta)$ is a random variable that takes values in $\mathbb{Z}_+^k$. Given an anonymous game $g \colon [k]^n \to [0, 1]^n$, and a player $i \in [n]$, $g_i(\cdot)$ is a function of a_i and $N(a_{-i})$; therefore, for any action $a_i \in [k]$, $g_i^\delta(a_i, a_{-i}) = E[f(N(a_{-i}^\delta))]$, for some function $f \colon \mathbb{Z}_+^k \to [0, 1]$. Since any such f can be realized by setting $g_i(a_i, a_{-i}) = f(N(a_{-i}))$, we have

$$\lambda(n, k, \delta) = \max_{f,a,b} E\left[f(N(a^\delta))\right] - E\left[f(N(b^\delta))\right],$$

where the maximum is over all $f \colon \mathbb{Z}_+^k \to [0, 1]$, and $a, b \in [k]^{n-1}$ subject to $\rho(a, b) = 1$. The maximum on the right-hand side is attained when f achieves the total variation distance between $N(a^\delta)$ and $N(b^\delta)$; therefore, by (arbitrarily) fixing the place in which a and b differ, we have

$$\begin{aligned} \lambda(n, k, \delta) &= \max_{a \in [k]^{n-2}} d_{TV}\left(e_1^\delta + N(a^\delta), e_2^\delta + N(a^\delta)\right) \\ &= (1 - \delta) \max_{a \in [k]^{n-2}} d_{TV}\left(e_1 + N(a^\delta), e_2 + N(a^\delta)\right). \end{aligned} \tag{4.1}$$

4.1 Proof of Theorem 1

In light of (4.1), the next lemma implies the upper bound of Theorem 1.

Lemma 8 *For every $k \geq 2$, $n \geq 1$, and $0 < \delta < 1$,*

$$\max_{a \in [k]^n} d_{TV}(e_1 + N(a^\delta), e_2 + N(a^\delta)) \leq P(S_n^{2\delta/k} \in \{0, 1\}).$$

Proof Let $a \in [k]^n$ be arbitrary. Let $X_1, \ldots, X_n \in \{e_1, \ldots, e_k\}$ be independent random vectors indicating the realizations of $a_1^\delta, \ldots, a_n^\delta$, respectively. Namely,

$$P(X_i = e_j) = \begin{cases} 1 - \delta + \frac{\delta}{k} & j = a_i, \\ \frac{\delta}{k} & j \neq a_i, \end{cases}$$

We would like to construct a coupling (Z_n, Z_n') such that $Z_n \sim e_1 + \sum_{i=1}^n X_i$, $Z_n' \sim e_2 + \sum_{i=1}^n X_i$ and $P(Z_n \neq Z_n') = P(S_n^{2\delta/k} \in \{0, 1\})$. To this end, we define random variables $X'_1, \ldots, X'_n$ that have the same joint distribution as $X_1, \ldots, X_n$, and let $Z_m = e_1 + \sum_{i=1}^m X_i$ and $Z'_m = e_2 + \sum_{i=1}^m X'_i$, for every $m = 1, \ldots, n$.

Loosely speaking, each X'_i is going to be defined inductively as a function of $X_1, \ldots, X_i$ in the following manner. In the event that $Z_{i-1} \neq Z'_{i-1}$, and $X_i \in \{e_1, e_2\}$, and X_i is a result of a perturbation (rather than due to a_i being in $\{1, 2\}$), then X'_i is defined as the elemnt of $\{e_1, e_2\} \setminus \{X_i\}$. In any other event, $X'_i := X_i$.

Formally, the random variables $X_1, \ldots, X_n$ are realized as follows:

$$X_i = \chi_i e_{U_i} + (1 - \chi_i) e_{a_i},$$

where $\chi_1, \ldots, \chi_n \sim \text{Bernoulli}(\delta)$, $U_1, \ldots, U_n \sim \text{uniform}([k])$ are all independent random variables.

The X'_is are coupled with the X_is through the following definition:

$$X'_i = \chi_i e_{U'_i} + (1 - \chi_i) e_{a_i},$$

where $U'_1, \ldots, U'_n$ are defined recursively by

$$U'_i = \begin{cases} 3 - U_i & Z_{i-1} \neq Z'_{i-1} \text{ and } U_i \in [2], \\ U_i & \text{otherwise}, \end{cases}$$

setting $Z_0 = e_1$, $Z'_0 = e_2$.

We explain why $X'_1, \ldots, X'_n$ are indeed independent random variables with $X'_i \sim a_i^\delta$ for every $i \in [n]$. Let $\mathcal{F}_i$ be the sigma-algebra generated by $\chi_1, U_1, \ldots, \chi_i, U_i$. By its definition, the distribution of U'_i is uniform in $[k]$ conditioned on $\mathcal{F}_{i-1}$, for every i; therefore $X'_i \sim a_i^\delta$ conditioned on $\mathcal{F}_{i-1}$. Furthermore, X'_i is $\mathcal{F}_i$-measurable ; therefore $X'_i \sim a_i^\delta$ conditioned on $X'_1, \ldots, X'_{i-1}$.

The definition of X'_i is such that $Z_i = Z'_i$ implies that $Z_{i+1} = Z'_{i+1}$, for every $i \in [n-1]$; therefore $Z_n = Z'_n$ iff there exists $i \in [n]$ such that $Z_i = Z'_i$. Furthermore, for every $0 \le i \le n$ and $3 \le j \le k$, $(Z_i)_j = (Z'_i)_j$ and $(Z_i)_1 + (Z_i)_2 = (Z'_i)_1 + (Z'_i)_2$; therefore $Z_n = Z'_n$ iff there exists $1 \le i \le n$ such that $(Z_i)_1 = (Z'_i)_1$.

Let $S_i := 1 - (Z_i)_1 + (Z'_i)_1$, $i = 0, \ldots, n$. Note that S_i is almost a symmetric random walk on $\mathbb{Z}$ except that it stays put forever once it hits 1. A direct calculation shows that conditioned on $S_i \neq 1$,

$$S_{i+1} = \begin{cases} S_i & \text{w.p.} \quad 1 - \frac{2\delta}{k}, \\ S_i + 1 & \text{w.p.} \quad \frac{\delta}{k}, \\ S_i - 1 & \text{w.p.} \quad \frac{\delta}{k}. \end{cases} \tag{4.2}$$

Since (4.2) is exactly the rule of $(S_i^{2\delta/k})_{i=0}^\infty$ (unlike S_i, $S_i^{2\delta/k}$ does not stop when it hits 1), Lemma 5 completes the proof of Lemma 8. □

The following lemma states that the upper bound of Lemma 8 is tight in the case of $k \ge 3$.

Lemma 9 *For every* $k \geq 3$, $n \geq 1$, *and* $0 < \delta < 1$,

$$\max_{a \in [k]^n} d_{TV}(e_1 + N(a^\delta), e_2 + N(a^\delta)) \geq P(S_n^{2\delta/k} \in \{0, 1\}).$$

Proof Consider the strategy profile $\bar{3} \in [k]^n$ in which all of the players take action 3. Let X be the random variable that counts the difference between the number of players who play 1 and those who play 2 under the mixed strategy profile $\bar{3}^\delta$. Formally, define $f \colon \mathbb{Z}^k \to \mathbb{Z}$ by $f(x_1, \ldots, x_n) = x_1 - x_2$. Then, $X := f(N(\bar{3}^\delta))$. Since $f(e_1 + N(\bar{3}^\delta)) = X + 1$ and $f(e_2 + N(\bar{3}^\delta)) = X - 1$, we have

$$\begin{aligned} d_{TV}(e_1 + N(\bar{3}^\delta), e_2 + N(\bar{3}^\delta)) &\geq d_{TV}(X + 1, X - 1) \\ &\geq P(X + 1 > 0) - P(X - 1 > 0) = P(X \in \{0, 1\}). \end{aligned}$$

The proof of Lemma 9 is complete since $X \sim S_n^{2\delta/k}$. □

4.2 Proof of Theorem 2

Let $X_1, X_2, \ldots$ be i.i.d. Bernoulli$(\delta/2)$ random variables. Define

$$M(n, \delta) = \max_{l, s \in \{0, \ldots, n\}} P\left(\sum_{i=0}^{l} X_i + \sum_{j=l+1}^{n} (1 - X_j) = s\right),$$

and $M(0, \delta) = 1$ by convention.

Lemma 10 *For every* $n \geq 2$ *and* $\delta \in (0, 1)$,

$$\lambda(n, 2, \delta) = (1 - \delta) M(n - 2, \delta).$$

Proof By (4.1) it is sufficient to prove that

$$\max_{a \in [2]^n} d_{TV}\left(e_1 + N(a^\delta), e_2 + N(a^\delta)\right) = M(n, \delta).$$

For every $a \in [2]^n$ there is an $l \in \{0, \ldots, n\}$ such that $N(a) = (l, n - l)$ and vice versa; therefore it is sufficient to prove that for each such pair $a \in [2]^n$ and $l \in \{0, \ldots, n\}$,

$$\begin{aligned} &d_{TV}\left(e_1 + N(a^\delta), e_2 + N(a^\delta)\right) \\ &\quad = \max_{s \in \{0, \ldots, n\}} P\left(\sum_{i=0}^{l} X_i + \sum_{j=l+1}^{n} (1 - X_j) = s\right). \end{aligned}$$

Let X be the random variable that counts the number of players who play 1 under the mixed strategy profile a^δ. Formally, X is defined by $N(a^\delta) = (X, n - X)$. Let

$f: x \mapsto n+1-x$. Since $e_1 + N(a^\delta) = (X+1, f(X+1))$ and $e_2 + N(a^\delta) = (X, f(X))$,

$$d_{TV}(e_1 + N(a^\delta), e_2 + N(a^\delta)) = d_{TV}(X+1, X).$$

Since X is PB, by Lemma 6,

$$d_{TV}(X+1, X) = \max_s P(X = s).$$

The proof is complete since $X \sim \sum_{i=0}^{l} X_i + \sum_{j=l+1}^{n}(1 - X_j)$. □

The next lemma states that the maximizers in the definition of $M(n, \delta)$ are $s = l = n/2$, for n even, and it provides upper and lower bounds, for n odd.

Lemma 11 *For every $n \in \mathbb{N}$ and $\delta \in (0, 1)$, let*

$$P_n = P\left(\sum_{i=1}^{n} X_i + \sum_{j=n+1}^{2n}(1 - X_j) = n\right).$$

Then,

$$P_{\lceil n/2 \rceil} \leq M(n, \delta) \leq \sqrt{P_{\lceil n/2 \rceil} P_{\lfloor n/2 \rfloor}}\,.$$

Proof To prove the first inequality, $P_{\lceil n/2 \rceil} \leq M(n, \delta)$, it is sufficient to show that $M(n, \delta)$ is decreasing in n, and $P_n \leq M(2n, \delta)$, for every $n \in \mathbb{N}$. The latter follows from the definition of $M(n, \delta)$ directly. The former holds since, there are some l, s_0, s_1, such that

$$\begin{aligned} M(n+1, \delta) = P(X_{n+1} = 0) P\left(\sum_{i=1}^{l} X_i + \sum_{j=l+1}^{n}(1 - X_j) = s_0\right) \\ + P(X_{n+1} = 1) P\left(\sum_{i=1}^{l} X_i + \sum_{j=l+1}^{n}(1 - X_j) = s_1\right) \leq M(n, \delta). \end{aligned}$$

It remains to prove the second inequality $M(n, \delta) \leq \sqrt{P_{\lceil n/2 \rceil} P_{\lfloor n/2 \rfloor}}$. Let l and s be such that $M(n, \delta) = P\left(\sum_{i=1}^{l} X_i + \sum_{j=l+1}^{n}(1 - X_j) = s\right)$. Define $\epsilon_1, \ldots, \epsilon_n \in \{+1, -1\}$ by $\epsilon_i = +1$ $(i \leq l)$ and $\epsilon_i = -1$ $(i > l)$. Let $Y = \sum_{i=1}^{\lceil n/2 \rceil} \epsilon_i X_i$ and $Z = \sum_{i=\lceil n/2 \rceil + 1}^{n} \epsilon_i X_i$. Then, by the Cauchy–Schwarz inequality,

$$\begin{aligned} M(n, \delta) = P(Y + Z = s - n) = \sum_t P(Y = t) P(Z = s - n - t) \\ \leq \sqrt{\sum_t (P(Y = t))^2 \sum_t (P(Z = t))^2}. \end{aligned}$$

The proof will be completed by showing that $\sum_t (P(Y=t))^2 = P_{\lceil n/2 \rceil}$ and $\sum_t (P(Z=t))^2 = P_{\lfloor n/2 \rfloor}$. More generally, we show that for every n and every $\epsilon_1, \ldots, \epsilon_n \in \{+1, -1\}$, letting $X = \sum_{i=1}^{n} \epsilon_i X_i$,

$$\sum_t (P\,(X=t))^2 = P_n. \tag{4.3}$$

Since

$$\sum_t (P\,(X=t))^2 = P(X=X'),$$

where X' is an independent copy of X, the case of $\epsilon_1 = \cdots = \epsilon_n$ is evident. It remains to show that toggling one of the ϵ_is does change the quantity on the left-hand side of (4.3). More generally, we show that for any two independent discrete random variables, X and Y,

$$P(X+Y=X'+Y') = P(X-Y=X'-Y'),$$

where X', Y' are independent copies of X, Y. This is true since,

$$P(X+Y=X'+Y') = P(X-Y'=X'-Y) = P(X-Y=X'-Y').$$

□

The proof of Theorem 2 follows immediately from Lemmata 10 and 11, since $S_n^{\delta(1-\delta/2)} \sim \sum_{i=1}^{n} (X_i - X_{n+i})$.

4.3 Proof of Theorem 3

By Theorem 2, the second part of Theorem 3, the case of $k=2$, follows from the next claim.

Claim 12 *For every $n \in \mathbb{N}$, and $0 < r \leq \frac{1}{2}$, let $\eta = 2\pi n r$. Then,*

$$\left| P\left(S_n^r = t\right) - \eta^{-\frac{1}{2}} \right| = \mathcal{O}(\eta^{-1}), \qquad t = 0, 1.$$

Proof Since $r \leq \frac{1}{2}$, the increments $S_{i+1}^r - S_i^r$ can be realized as the difference of two i.i.d. Bernoulli random variables; therefore S_n^r is a Poisson binomial random variable. Applying Lemma 7 with $\mu = E[S_n^r] = 0$, $\sigma^2 = Var[S_n^r] = nr$ gives

$$\left| \sigma P(S_n^r = t) - \frac{1}{\sqrt{2\pi}} \right| = \mathcal{O}(\sigma^{-1}), \qquad t = 0, 1,$$

which completes the proof of Lemma 12. □

For the first part of Theorem 3, the case of $k \geq 3$, we have to consider $r > \frac{1}{2}$ as well.

Claim 13 *For every $n \in \mathbb{N}$ and $0 < r \leq 1$, let $\eta = \frac{1}{2}\pi nr$. Then,*

$$\left| P(S_n^r \in \{0,1\}) - \eta^{-\frac{1}{2}} \right| = \mathcal{O}(\eta^{-1}).$$

Proof The case of $r \leq \frac{1}{2}$ is an immediate consequence of Claim 12. The case of $r = 1$ and n even, too, follows from Claim 12, since $S_{2m}^1 \sim 2S_m^{\frac{1}{2}}$, and $P(S_{2m}^1 = 1) = 0$. By Lemma 5, $P(S_n^1 \in \{0,1\})$ is monotonic in n; therefore the claim holds for $r = 1$ and n odd, as well.

For $r \in \left(\frac{1}{2}, 1\right)$, let us realize S_n^r as S_X^1, where $X \sim \text{binomial}(n, r)$ independently of $(S_i^1)_{i=1}^n$. Let C_1 be the constant of the "$\mathcal{O}$" term in the claim for the case of $r = 1$. Let $f(x,r) = (1 + x/r)^{-\frac{1}{2}}$, and $C_2 = \max\{|\mathrm{d}f/\mathrm{d}x| : \frac{1}{2} \leq r \leq 1,\ |x| \leq \frac{1}{4}\}$. The proof of Claim 13, and Theorem 3 thereby, is completed as follows:

$$\begin{aligned}
\left| P(S_n^r \in \{0,1\}) - \eta^{-\frac{1}{2}} \right| &= \left| P(S_X^1 \in \{0,1\}) - \eta^{-\frac{1}{2}} \right| \\
&\leq P(|X - nr| > n/4) \\
&\quad + \sum_{k:|k-nr| \leq n/4} P(X = k) \left(\left| P(S_k^1 \in \{0,1\}) - \left(\tfrac{1}{2}\pi k\right)^{-\frac{1}{2}} \right| \right. \\
&\quad \left. + \left| \left(\tfrac{1}{2}\pi k\right)^{-\frac{1}{2}} - \left(\tfrac{1}{2}\pi nr\right)^{-\frac{1}{2}} \right| \right) \\
&\leq \frac{16Var[X]}{n^2} \\
&\quad + \sum_{k:|k-nr| \leq n/4} P(X = k) \left(C_1 \frac{2}{\pi k} + \eta^{-\frac{1}{2}} \left| f(k/n - r, r) - f(0, r) \right| \right) \\
&\leq \frac{4}{n} + \frac{C_1 8}{\pi n} + \eta^{-\frac{1}{2}} C_2 \sum_{k:|k-nr| \leq n/4} P(X = k)\, |k/n - r| \\
&\leq \mathcal{O}(n^{-1}) + C_2 \eta^{-\frac{1}{2}} E \left| \frac{X}{n} - r \right| \\
&\leq \mathcal{O}(n^{-1}) + C_2 \eta^{-\frac{1}{2}} \sqrt{Var\left[\frac{X}{n}\right]} = \mathcal{O}(n^{-1}).
\end{aligned}$$

The second and the last inequalities use Chebyshev's and Jensen's inequalities, respectively, and $Var[X] = nr(1-r) \leq n/4$. □

4.4 Proof of Theorem 4

Let g be an n-player k-action anonymous game. Every ϵ-Nash equilibrium in g^δ translates to a $(\delta + \epsilon)$-Nash equilibrium in g by identifying the actions of g^δ with δ-peturbed actions in g; therefore the first part of Theorem 4 is an immediate consequence of the following theorem.

Theorem 14 (Theorem 6.1 in Azrieli and Shmaya 2013) *Any n-player k-action λ-Lipschitz game admits a $2k\lambda$-Nash equilibrium in pure strategies.*

The second part of Theorem 4 follows from Theorem 3, and setting $\delta = \lambda(n, k, \delta)$, and $\epsilon = 2\delta$.

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